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Two-parameter extended Hubbard Hamiltonian with gl(2, 1) supersymmetry*

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Abstract. We investigate under which circumstances extended Hubbard models, including bond-charge, exchange, and pair-hopping terms, are invariant under $gl(2,1)$ superalgebra. This happens for a two-parameter Hamiltonian which includes as particular cases the $t{-}J$, the EKS and the one-parameter BGLZ Hamiltonians, all integrable in one dimension. We show that the two parameter Hamiltonian can be recasted as the sum of the BGLZ Hamiltonian plus the graded permutation operator of electronic states on neighbouring sites. The integrability of the corresponding one-dimensional model is discussed.

PACS. 71.27.+a Strongly correlated electron systems; heavy fermions – 75.10.Lp Band and itinerant models

1 Introduction

Recently the interest in low-dimensional correlated electron systems has motivated some attention on certain integrable one-dimensional models which happen to be invariant under the graded algebra $gl(2,1)$ [1–7]. More precisely, starting from R-matrices which can be built from representations of $gl(2,1)$, different integrable interacting electron models are obtained in the Hamiltonian limit of the transfer matrix thereof constructed. In particular, from the family of four-dimensional representations $[b,1/2]$ parametrized by a parameter $b \neq \pm 1/2$ (see below), a one-parameter extended Hubbard model has been derived [2] (BGLZ), and successively its eigenvalues have been evaluated within quantum inverse scattering method (QISM) [8] by means of algebraic Bethe Ansatz [5]. Nevertheless, while the machinery for constructing integrable electron models with given symmetry starting from R-matrices built in the symmetry algebra is well established within the framework of QISM, little is known about the inverse problem, namely the question of integrability of Hamiltonians with given (super)symmetry algebra. With this in mind, in the present paper we begin to derive the more general extended Hubbard Hamiltonian invariant under the superalgebra $gl(2,1)$. This turns out to be (in any dimension) a two-parameter Hamiltonian, which contains as limiting cases the one-parameter BGLZ, the $t{-}J$ [1] and EKS [9] models. By successively expressing such Hamiltonian as a sum of local two-site Hamiltonians, by means of projection operators in the representation $[b,1/2]\otimes[b,1/2]$ of $gl(2,1)$ we then show that infact the whole class of $gl(2,1)$ invariant Hamiltonians can be written as the sum of the BGLZ Hamiltonian and the graded permutation operator of neighbouring electronic states (which coincides with the EKS model Hamiltonian), commuting at the local level. The integrability of the resulting model in one dimension is discussed.

2 Two-parameters supersymmetric Hamiltonian

The extended Hubbard model was originally derived by Hubbard himself [10]. Its Hamiltonian reads

$$
\begin{aligned}
H = &-\sum_{\langle \mathbf{j},\mathbf{k}\rangle}\sum_{\sigma}[t - X(n_{\mathbf{j},-\sigma}+n_{\mathbf{k},-\sigma}) \\
&+\tilde{X} n_{\mathbf{j},-\sigma}n_{\mathbf{k},-\sigma}]c^{\dagger}_{\mathbf{j},\sigma}c_{\mathbf{k},\sigma} + U\sum_{\mathbf{j}} n_{\mathbf{j},\uparrow}n_{\mathbf{j},\downarrow} \\
&+\frac{V}{2}\sum_{\langle \mathbf{j},\mathbf{k}\rangle} n_{\mathbf{j}}n_{\mathbf{k}} + \frac{W}{2}\sum_{\langle \mathbf{j},\mathbf{k}\rangle}\sum_{\sigma,\sigma'} c^{\dagger}_{\mathbf{j},\sigma}c^{\dagger}_{\mathbf{k},\sigma'}c_{\mathbf{j},\sigma'}c_{\mathbf{k},\sigma} \\
&+\frac{Y}{2}\sum_{\langle \mathbf{j},\mathbf{k}\rangle}\sum_{\sigma} c^{\dagger}_{\mathbf{j},\sigma}c^{\dagger}_{\mathbf{j},-\sigma}c_{\mathbf{k},-\sigma}c_{\mathbf{k},\sigma} + \mu_e\sum_{\mathbf{j},\sigma} n_{\mathbf{j},\sigma},
\end{aligned} \quad (1)
$$

where $c^{\dagger}_{\mathbf{j},\sigma}, c_{\mathbf{j},\sigma}$ are fermionic creation and annihilation operators ($\{c_{\mathbf{j},\sigma'}, c_{\mathbf{k},\sigma}\} = 0,\ \{c_{\mathbf{j},\sigma}, c^{\dagger}_{\mathbf{k},\sigma'}\} = \delta_{\mathbf{j},\mathbf{k}}\,\delta_{\sigma,\sigma'}\mathbb{I}$, $n_{\mathbf{j},\sigma} \doteq c^{\dagger}_{\mathbf{j},\sigma}c_{\mathbf{j},\sigma}$, $n_{\mathbf{j}} = \sum_{\sigma} n_{\mathbf{j},\sigma}$) on a d-dimensional lattice Λ ($\mathbf{j},\ \mathbf{k}\in\Lambda$, $\sigma\in\{\Uparrow,\Downarrow\}$), and $\langle\mathbf{j},\ \mathbf{k}\rangle$ stands for nearest neighbors ($n.n.$) in Λ. In (1) the first term represents the band energy of the electrons, and the remaining terms describe their Coulomb interaction energy in a narrow band

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approximation: U parametrizes the on-site diagonal interaction, V the neighboring site charge interaction, X the bond-charge interaction, W the exchange term, and Y the pair-hopping term. An explicit evaluation of the relative size of these contributions – all generated from onsite and *n.n.* matrix elements of the Coulomb interaction – was already given in [10]. For the sake of generality, an additional coupling term $\tilde{X}$ has been included, which correlates hopping with on-site occupation number. Finally, μ_e is the chemical potential.

The circumstances under which (1) is invariant for the $su(2)\bigoplus su(2)$ symmetry algebra of the standard Hubbard model were already discussed in [11], were such symmetry was used to construct explicitly ordered ground states for (1) in particular regions of the phase-space. Here we investigate instead which relations the coupling constants have to satisfy in order to have H invariant under the superalgebra $gl(2,1)$. The latter is generated by the identity and the generators of $osp(2,2)$ algebra, namely

$$S_+ = \sum_{\mathbf{j}} c^\dagger_{\mathbf{j},\uparrow} c_{\mathbf{j},\downarrow},\ S_- = \sum_{\mathbf{j}} c^\dagger_{\mathbf{j},\downarrow} c_{\mathbf{j},\uparrow},$$
$$S_z = \frac{1}{2}\sum_{\mathbf{j}} (n_{\mathbf{j},\uparrow} - n_{\mathbf{j},\downarrow}),\ B = \frac{1}{2}\sum_{\mathbf{j}} (n_{\mathbf{j},\uparrow} + n_{\mathbf{j},\downarrow} - 1) \quad (2)$$

in the even part, and

$$Q_\sigma = \sum_{\mathbf{j}} s^{|\mathbf{j}|} \left[(1 - n_{\mathbf{j},-\sigma}) c_{\mathbf{j},\sigma} + b'\, n_{\mathbf{j},-\sigma} c_{\mathbf{j},\sigma}\right], \quad (3)$$

$Q^\dagger_\sigma$ in the odd part (for their commutation relations, see for instance [3]). Here $s = \pm 1$ and $b' \in \mathbf{C}$ are free. Since the properties of Hamiltonian (1) are not changed by adding a constant term, we will concentrate on the invariance of H under $osp(2,2)$. A straightforward calculation shows that H in fact commutes with all the eight generators above if, and only if, the following constraint equations are fulfilled by the Hamiltonian parameters,

$$sX = st - b'\big(st + \frac{U}{Z}\big), \quad s\tilde{X} = (1-b')^2\big(st + \frac{U}{Z}\big),$$
$$W = -\Big[st - {b'}^2\big(st + \frac{U}{Z}\big)\Big] = V, \quad Y = \frac{U}{Z}, \quad \frac{\mu_e}{Z} = st, \quad (4)$$

with Z number of *n.n.* in Λ. It is easily verified that in fact the freedom in the choice of s corresponds to the possibility of changing the sign of all the contributions t, X, and $\tilde{X}$ to the hopping term in (1), by means of the canonical transformation $c_{\mathbf{j},\sigma} \to (-)^{|\mathbf{j}|} c_{\mathbf{j},\sigma}$. As the latter leaves unchanged the remaining part of the Hamiltonian, from now on we choose $s = +1$.

In the following, we shall label by H_{ss} the Hamiltonian (1) in which the parameters are constrained by (4). Apart from the hopping amplitude t, which can be scaled away, the independent free parameters in (4) are two, b' and U/Z. Setting $W = 0$ fixes one of the two parameters and the one-parameter BGLZ Hamiltonian discussed in [2] is recovered, whereas setting $b' = 0 = U/Z$, no more free parameter is left, and the integrable supersymmetric $t{-}J$ model is obtained. Moreover, for $b' = 1$ the other parameter U/Z becomes a multiplicative constant, and the EKS Hamiltonian is obtained [9]. Notice that the latter, however, turns out to have a larger symmetry superalgebra, *i.e.* $u(2,2)$, of which $gl(2,1)$ is a subalgebra. As all of these particular cases are integrable in one dimension, one may wonder whether the whole two-parameter class of $gl(2,1)$ invariant Hamiltonians is integrable for $d = 1$.

Let us notice that the fact that Hamiltonian H_{ss} commutes with the operators given in (2, 3), allows the construction of many eigenstates of H. Indeed, starting form a known eigenstate $|\psi\rangle$ of H_{ss}, which for simplicity we assume to be of highest weight, $S^n_-|\psi\rangle$ (with $n \leq N_e$, and N_e number of electrons on the lattice) and $Q_\sigma|\psi\rangle$ are still eigenstates of H_{ss}. In particular, $Q_\sigma|\psi\rangle$ generate eigenstates with holes with respect to $|\psi\rangle$. For instance, by taking $|\psi\rangle$ equal to the fully polarized ferromagnetic state (with up spins) at half-filling, the state $Q_\downarrow|\psi\rangle$ can be shown to be the ground state in appropriate regions of the parameter space [12] with one hole in a half-filled band. In general, the eigenstates generated in this way are degenerate in energy with $|\psi\rangle$ for $\mu_e/Z = t$, whereas they could have lower energies than $|\psi\rangle$ for different filling (and chemical potential) values.

3 R-matrix in gl(2, 1) and the BGLZ model

In order to investigate the relation between Hamiltonian (1) in $d = 1$ and integrable models in $gl(2,1)$, we briefly review here the construction of the R-matrix acting on the tensor product of two four-dimensional representations $[b, \frac{1}{2}]$ of $gl(2,1)$, according to the scheme developed in [5]. For a more detailed exposition of the general scheme of QISM see also [8]. The origin of the parameter b is in the $u(1)$ subalgebra of the even part of $gl(2,1)$, which – in the fermionic representation given above – is generated by B. The even part consists as well of a $su(2)$ algebra generated by $S_\pm$, S_z. Thus the basis for the four dimensional representation can be labeled also by the eigenvalues of the total spin and the z component of the spin. The states can be ordered in the following way

$$|b, \frac{1}{2}, \frac{1}{2}\rangle \equiv |1\rangle, \quad |b, \frac{1}{2}, -\frac{1}{2}\rangle \equiv |2\rangle,$$
$$|b - \frac{1}{2}, 0, 0\rangle \equiv |3\rangle, \quad |b + \frac{1}{2}, 0, 0\rangle \equiv |4\rangle, \quad (5)$$

where $|1\rangle = (1, 0, 0, 0)$ and analogous definitions hold for $|2\rangle$, $|3\rangle$, and $|4\rangle$. In the electronic model given by (1) they can be identified with the four possible electronic states at a given lattice site, *i.e.* a single electron up, a single electron down, an empty site and a doubly occupied site respectively. As we are in a $\mathbb{Z}_2$ graded algebra, the basis vectors have two possible gradings. The first two states have grading equal to 1, the other two equal to 0. It was shown in [2,5], that in the above basis a R-matrix satisfying Yang-Baxter equations

$$R^b_{1,2}(\lambda-\mu) R^b_{1,3}(\lambda) R^b_{2,3}(\mu) = R^b_{2,3}(\mu) R^b_{1,3}(\lambda) R^b_{1,2}(\lambda-\mu), \quad (6)$$

(where the subindices refer to the spaces on which R acts nontrivially) reads

$$R^b(\mu) = \frac{2\mu-2b+1}{2\mu+2b-1} It_1 + \frac{2\mu-2b+1}{2\mu+2b-1}\frac{2\mu-2b-1}{2\mu+2b+1} It_2 + It_3. \qquad (7)$$

Here It_1, It_2, and It_3 are the intertwiners for the three irreducible components of the tensorproduct $[b,\frac{1}{2}]\otimes[b,\frac{1}{2}]$, as derived explicitly in [5], and reported here in the Appendix. Notice that by construction the three intertwiners, as well as the R-matrix, are all invariant under $gl(2,1)$. $\mu=0$ is the shift point, *i.e.* $R^b(0)=P$, where P is the graded permutation operator of the states (5) at sites j and k. Explicitly

$$P\,|n\rangle_j\otimes|m\rangle_k = |m\rangle_k\otimes|n\rangle_j = (-)^{\epsilon_n\epsilon_m}|n\rangle_j\otimes|m\rangle_k, \quad n,m=1,\ldots,4 \qquad (8)$$

where ϵ_m is the grading of the state $|m\rangle$. It is easily verified that $P = It_3 + It_2 - It_1$.

The existence of a shift point guarantees the possibility of constructing from R-matrices integrable local one-dimensional models. Here by local we mean that the model Hamiltonian $\mathcal{H}$ can be written as the sum $\sum_{j=1}^{N}\mathcal{H}_j$ of contributions $\mathcal{H}_j$ which act non-trivially only on j and a finite number of neighbours of j. The number of commuting local Hamiltonians is infinite in the thermodynamical limit. They can be identified with the integrals of motion, and enter as coefficients (H_k) in the power expansion in μ of the logarithm of the transfer matrix $\tau(\mu)$,

$$\ln\tau(\mu)\tau(0)^{-1} = \sum_{k=1}^{\infty}\frac{\mu^k}{k!}H_{k+1}. \qquad (9)$$

Here $\tau(\mu)$ is the supertrace of the monodromy matrix, which in turn is obtained from the R-matrices by taking matrix products in one component of the tensor product (the auxiliary space).

The first integral of motion generated in this way has been shown to coincide (apart from a multiplicative constant) with the BGLZ Hamiltonian, as can be verified by comparison of

$$H_2 = \left(\frac{\partial\ln\tau(\mu)}{\partial\mu}\right)_{\mu=0} = \sum_j P_{j,j+1}\left(\frac{\partial R_{j,j+1}(\mu)}{\partial\mu}\right)_{\mu=0}, \qquad (10)$$

with (1), (4) for $W=0$. This proves integrability of the BGLZ one-parameter supersymmetric Hamiltonian.

4 An alternative derivation of supersymmetry condition

The fact that the most general Hamiltonian invariant under $gl(2,1)$ has two free parameters is related to the number of independent intertwiners in the representation, which is two (since $It_1+It_2+It_3=\mathbb{I}$, the latter being the identity matrix). One may then wonder which is the actual expression of H_{ss} in terms of the intertwiners. A tedious but straightforward calculation allows us to recognize that the following relation holds,

$$H_{ss} = k\left[H_{BGLZ} + a\sum_j\left(P_{j,j+1}-\mathbb{I}\right)\right], \qquad (11)$$

where $k=-[2/(2b-1)+a]^{-1}$. The Hamiltonian parameters, written in terms of the arbitrary constants a, and b, are given by

$$\frac{\mu_e}{\mathbf{k}Z} = k^{-1}, \quad \frac{U}{Z} = k\left(\frac{4}{4b^2-1}+a\right) = Y,$$
$$W = -kaX = 1-\frac{2k}{\sqrt{4b^2-1}},$$
$$\tilde{X} = 1-k\left(a+\frac{4}{\sqrt{4b^2-1}}-\frac{2}{1+2b}\right), \qquad (12)$$

which are easily checked to satisfy (4) for $b' = -\sqrt{\frac{b+\frac{1}{2}}{b-\frac{1}{2}}}$.

The above expression (11) is particularly interesting, in that it shows that all the Hamiltonians invariant under $gl(2,1)$ are sum of two separatly integrable models, the BGLZ and the EKS. In fact setting $a=a'/k$ and taking $k=0$, $a'\neq 0$, the model obtained is nothing but the EKS Hamiltonian [9], even though such Hamiltonian cannot be obtained as derivative from the R-matrix (7) for any suitable value of b. Indeed, the model in this case, as it simply acts as a graded permutation of the electronic states at sites j and $j+1$, is integrable [13,14]. This observation suggests that there must exist a more general R-matrix than (7) which satisfies YBE. As a further example, let us notice that a different class of one-parameter supersymmetric Hamiltonian is obtained in [3], starting from a slightly different one-parameter R-matrix in $gl(2,1)$. There the main difference, from the physical point of view, is that the resulting model has vanishing coefficient of the pair-hopping term (Y), and non-vanishing coefficient of the exchange term (W), as opposite to the case described by BGLZ Hamiltonian.

A more general R-matrix still satisfying YBE is the trigonometric R-matrix which can be built in the deformed algebra $\mathcal{U}_q(sl(2,1))$. In this case the R-matrix acquires a natural extra parameter which is the deformation parameter q. Such two-parameter R-matrix, and the corresponding one-dimensional integrable models, have also been considered in the literature [3,6,7,15]. However in particular the resulting electronic models are definitly different from the one described by H_{ss}, in that they exhibit anisotropy in the correlated hopping terms.

The possibility that the Hamiltonian H_{ss} is itself integrable, in the sense that it can be obtained directly

$$
It_3 = \frac{1}{2}\begin{pmatrix}
0 &&&&&&&&&&&&&&& \\
& \bar{\beta} &&& -\bar{\beta} &&&&&&& -\sqrt{\beta\bar{\beta}} &&& -\sqrt{\beta\bar{\beta}} & \\
&& 1 &&&&&& 1 &&&&&&& \\
&&& 0 &&&&&&&&&&&& \\
& -\bar{\beta} &&& \bar{\beta} &&&&&&& \sqrt{\beta\bar{\beta}} &&& \sqrt{\beta\bar{\beta}} & \\
&&&&& 0 &&&&&&&&&& \\
&&&&&& 1 &&& 1 &&&&&& \\
&&&&&&& 0 &&&&&&&& \\
&& 1 &&&&&& 1 &&&&&&& \\
&&&&&& 1 &&& 1 &&&&&& \\
&&&&&&&&&& 0 &&&&& \\
& -\sqrt{\beta\bar{\beta}} &&& \sqrt{\beta\bar{\beta}} &&&&&&& \beta &&& \beta & \\
&&&&&&&&&&&& 0 &&& \\
&&&&&&&&&&&&& 0 && \\
& -\sqrt{\beta\bar{\beta}} &&& \sqrt{\beta\bar{\beta}} &&&&&&& \beta &&& \beta & \\
&&&&&&&&&&&&&&& 0
\end{pmatrix} \tag{A.1}
$$

as derivative of some rational two-parameter R-matrix, is also hinted to by the fact that $P_{j,j+1}$ commutes with all the derivatives of $R^b_{j,j+1}$, which enter the definition of the infinite set of integrals of motion, and in particular of H_{BGLZ} through (10). Nevertheless, the definite prove would be to show that some linear combination of $R^b(\mu)$ and $P-\mathbb{I}$, which separately satisfy YBE, does satisfy YBE as well.

5 Conclusions

We have derived the most general extended Hubbard Hamiltonian invariant under the graded algebra $gl(2,1)$, which turns out to have two free parameters. The knowledge of this symmetry (which holds in any dimension) is useful to generate new eigenstates for the model. The latter, by means of known techniques [11,12,16], could be eventually proved to be the ground state in certain regions of phase space, also away from half-filling. Moreover, by successively confining our analysis to the one-dimensional case, we have shown that the supersymmetry condition for the Hamiltonian can be reformulated in terms of intertwining operators of the representation in such a way that H_{ss} is recognized as the sum of the BGLZ Hamiltonian plus an extra-term proportional to the graded permutation operator of electronic states on neighbouring sites. Work is in progress on the open point of deriving from such information the possible integrability of H_{ss}.

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Appendix

The projection operators $It_1 = \frac{1}{2}(\mathbb{I}-P)$, $It_2 = \frac{1}{2}(\mathbb{I}+P) - It_3$, and It_3 are given in [5]. Here we give It_3, from which the other two are easily obtained.

See equation (A.1) above

with $\beta = (2b+1)/4b$, $\bar{\beta} = (2b-1)/4b$.

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